Quantum limits in interferometric detection of gravitational radiation

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Introduction

What is Done: A spectral analysis of quantum noise in an optical interferometer was performed to detect gravitational radiation. The investigation focused on the Standard Quantum Limit (SQL) for that system.
The prominent aspect: Two different methods of beating the standard quantum limit are examined.

Gravitational Waves

- Gravitational waves are ripples in spacetime caused by the acceleration of massive objects, as predicted by Einstein's General Theory of Relativity.
- As a gravitational wave passes an observer, that observer will find spacetime distorted by the effects of strain. Distances between objects increase and decrease rhythmically as the wave passes, at a frequency equal to that of the wave.
- They can be detected with optical interferometer.

https://www.youtube.com/watch?v=DfVI6sBnu7Ylist=LLindex=1

Interferometer System

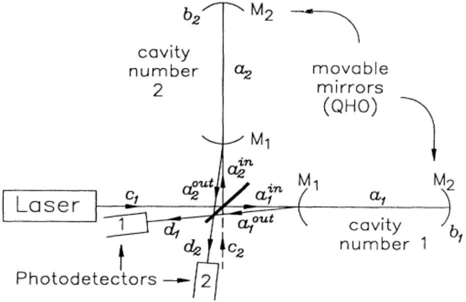


FIG. 1. The operator notation used for the interferometer system.

The Roadmap

- Derive Hamiltonian for a single arm of the interferometer.
- Derive the linearized quantum Langevin equations for the internal operators.
- Combine the output fields from each arm of the interferometer.
- Relate output fields to intensity in the optical detectors, define signal and noise based on this intensity fluctuations.
- Derive Standart Quantum Limit (SQL).

Hamiltonian for Single Arm

$$H_{\text{tot}} = H_{\text{sys}} + H_b + H_{\text{int}} \tag{1}$$

$$H_{\text{sys}} = \hbar\Delta a^\dagger a + \hbar\Omega b^\dagger b + \hbar\kappa a^\dagger a(b + b^\dagger) + \hbar k s(t)(b + b^\dagger). \tag{2}$$

Langevin Equations For Internal Operators

$$\begin{aligned}
\dot{a} &= -\frac{i}{\hbar}[a, H_{\text{sys}}] - \frac{\gamma_a}{2}a + \gamma_a^{1/2}a^{\text{in}} \\
&= -i\left[\Delta a + \kappa a(b + b^\dagger)\right] - \frac{\gamma_a}{2}a + \gamma_a^{1/2}a^{\text{in}}, \qquad (3) \\
\dot{b} &= -\frac{i}{\hbar}[b, H_{\text{sys}}] - \frac{\gamma_b}{2}b + \gamma_b^{1/2}b^{\text{in}} \\
&= -i\left[\Omega b + \kappa a^\dagger a + k s(t)\right] - \frac{\gamma_b}{2}b + \gamma_b^{1/2}b^{\text{in}}. \qquad (4)
\end{aligned}$$

Linearized Equation of Motions

$$\frac{d}{dt}\begin{pmatrix}\delta a\\ \delta a^\dagger\\ \delta b\\ \delta b^\dagger\end{pmatrix} = -\begin{pmatrix}\frac{\gamma_a}{2} & 0 & i\kappa & i\kappa\\ 0 & \frac{\gamma_a}{2} & -i\kappa^* & -i\kappa^*\\ i\kappa^* & i\kappa & \frac{\gamma_b}{2}+i\Omega & 0\\ -i\kappa^* & -i\kappa & 0 & \frac{\gamma_b}{2}-i\Omega\end{pmatrix}\begin{pmatrix}\delta a\\ \delta a^\dagger\\ \delta b\\ \delta b^\dagger\end{pmatrix} + \begin{pmatrix}\gamma_a^{1/2}\delta a^{\mathrm{in}}\\ \gamma_a^{1/2}\delta a^{\mathrm{in}\dagger}\\ \gamma_b^{1/2}\delta b^{\mathrm{in}}\\ \gamma_b^{1/2}\delta b^{\mathrm{in}\dagger}\end{pmatrix} + \begin{pmatrix}0\\ 0\\ -iks(t)\\ +iks(t)\end{pmatrix} \tag{5}$$

$$\frac{d}{dt}\delta a(t) = -A\delta a(t) + F(t) + g(t) \tag{6}$$

Solutions

$$\delta a(\omega) = (A - i\omega I)^{-1}\Big[F(\omega) + g(\omega)\Big], \tag{7}$$

$$\text{where} \quad F(\omega) = \begin{pmatrix} \gamma_a^{1/2}\delta a^{\mathrm{in}}(\omega) \\ \gamma_a^{1/2}\delta a^{\mathrm{in}\dagger}(\omega) \\ \gamma_b^{1/2}\delta b^{\mathrm{in}}(\omega) \\ \gamma_b^{1/2}\delta b^{\mathrm{in}\dagger}(\omega) \end{pmatrix}, \quad g(\omega) = iks(\omega)\begin{pmatrix} 0 \\ 0 \\ -1 \\ +1 \end{pmatrix}, \tag{8}$$

$$\text{and} \quad (A - i\omega I)^{-1} = \begin{pmatrix} \frac{\Lambda_1\Lambda_3\Lambda_4 + 2i\Omega\kappa^2|\alpha|^2}{\Lambda_1\Lambda_3\Lambda_4} & \frac{2i\Omega\kappa^2|\alpha|^2}{\Lambda_1\Lambda_3\Lambda_4} & \frac{-i\kappa}{\Lambda_1\Lambda_3} & \frac{-i\kappa}{\Lambda_1\Lambda_4} \\ \frac{-2i\Omega\kappa^2|\alpha|^2}{\Lambda_1\Lambda_3\Lambda_4} & \frac{\Lambda_1\Lambda_3\Lambda_4 - 2i\Omega\kappa^2|\alpha|^2}{\Lambda_1\Lambda_3\Lambda_4} & \frac{i\kappa^*}{\Lambda_1\Lambda_3} & \frac{i\kappa^*}{\Lambda_1\Lambda_4} \\ \frac{-i\kappa^*}{\Lambda_1\Lambda_3} & \frac{i\kappa^*}{\Lambda_1\Lambda_3} & \frac{1}{\Lambda_3} & 0 \\ \frac{i\kappa^*}{\Lambda_1\Lambda_4} & \frac{i\kappa^*}{\Lambda_1\Lambda_4} & 0 & \frac{1}{\Lambda_4} \end{pmatrix}. \tag{9}$$

Solutions

$$\Lambda_1 \equiv \Lambda_1(\omega) = \frac{\gamma_a}{2} - i\omega, \tag{10}$$

$$\Lambda_3 \equiv \Lambda_3(\omega) = \frac{\gamma_b}{2} + i(\Omega - \omega), \tag{11}$$

$$\Lambda_4 \equiv \Lambda_4(\omega) = \frac{\gamma_b}{2} - i(\Omega + \omega). \tag{12}$$

$$b_{ij}(\omega) = \left[\left(\mathbf{A} - i\omega\mathbf{I} \right)^{-1} \right]_{ij} \tag{13}$$

Recombining The Two Arms

$$\begin{pmatrix} \delta a_1^{\text{out}}(\omega) \\ \delta a_1^{\text{out}\dagger}(\omega) \\ \delta a_2^{\text{out}}(\omega) \\ \delta a_2^{\text{out}\dagger}(\omega) \end{pmatrix} = \gamma_a^{1/2} \begin{pmatrix} \delta a_1(\omega) \\ \delta a_1^{\dagger}(\omega) \\ \delta a_2(\omega) \\ \delta a_2^{\dagger}(\omega) \end{pmatrix} - \begin{pmatrix} \delta a_1^{\text{in}}(\omega) \\ \delta a_1^{\text{in}\dagger}(\omega) \\ \delta a_2^{\text{in}}(\omega) \\ \delta a_2^{\text{in}\dagger}(\omega) \end{pmatrix}, \tag{14}$$

$$\delta a_{12}^{\text{out}}(\omega) = \gamma_a^{1/2} \delta a_{12}(\omega) - \delta a_{12}^{\text{in}}(\omega). \tag{15}$$

Recombining The Two Arms

$$\delta a_i^{\mathsf{in}}(\omega) = \begin{pmatrix} \delta a_i^{\mathsf{in}}(\omega) \\ \delta a_i^{\mathsf{in}\dagger}(\omega) \\ \delta b_i^{\mathsf{in}}(\omega) \\ \delta b_i^{\mathsf{in}\dagger}(\omega) \end{pmatrix}. \tag{16}$$

$$\delta a_{12}^{\mathsf{out}}(\omega) = \mathbf{M}_{10}(\omega)\delta a_1^{\mathsf{in}}(\omega) + \mathbf{M}_{02}(\omega)\delta a_2^{\mathsf{in}}(\omega) + \mathbf{g}(\omega). \tag{17}$$

$$\mathbf{g}(\omega) = \gamma_a^{1/2} k S(\omega) \begin{bmatrix} i\left[b_{14}(\omega) - b_{13}(\omega)\right] \\ i\left[b_{24}(\omega) - b_{23}(\omega)\right] \\ \left[b_{14}(\omega) - b_{13}(\omega)\right] \\ -\left[b_{24}(\omega) - b_{23}(\omega)\right] \end{bmatrix}$$

Signal and Variance

$$\mathcal{S}(\omega) = \langle I_1(\omega) - I_2(\omega) \rangle$$

$$\mathcal{V} = \langle I_1(\omega) - I_2(\omega), I_1(\omega') - I_2(\omega') \rangle$$

$$a_1^{\text{out}}(t) = \left(I_1^{\text{out}}\right)^{1/2} e^{i\phi_1},$$
$$a_2^{\text{out}}(t) = \left(I_2^{\text{out}}\right)^{1/2} e^{i\phi_2},$$

$$I_1(t) - I_2(t) = d_1^\dagger d_1 - d_2^\dagger d_2 = 2\left(I_1^{\text{out}} I_2^{\text{out}}\right)^{1/2} \sin(\phi_1 - \phi_2) \approx 2I \sin(\phi_1 - \phi_2),$$
$$= 2I \sin(\phi_1 + \phi - \phi_2) = 2I \sin(\delta\phi_1 - \delta\phi_2) \approx 2I(\delta\phi_1 - \delta\phi_2),$$

Signal and Variance

$$\delta\phi_1 \approx \frac{X_2}{\langle X_1 \rangle} = \frac{i}{2\sqrt{I}} \left(\delta a_1^{\text{out}\dagger} - \delta a_1^{\text{out}} \right),$$

$$\delta\phi_2 \approx -\frac{X_1}{\langle X_2 \rangle} = -\frac{1}{2\sqrt{I}} \left(\delta a_2^{\text{out}\dagger} + \delta a_2^{\text{out}} \right)$$

$$\mathcal{S} = \frac{32 h \omega_g^2 \omega_0 s(\omega) I}{\Lambda_1(\omega)\Lambda_3(\omega)\Lambda_4(\omega)}, \quad \mathcal{S}(\omega) = f_s(\omega) h s(\omega)$$

$$\mathcal{V}(\omega) = f_v(\omega)\delta(\omega + \omega'), \quad \Delta h = \frac{\left[f_v(\omega_g) \right]^{1/2}}{|f_s(\omega_g)|} \frac{1}{\sqrt{\tau/2}}$$

Introducing coherent states (e.g., laser input) as input light fields

$$\mathcal{V} = 2I\left[1 + \frac{16\kappa^2\gamma_b\left[(\gamma_b/2)^2+\Omega^2+\omega^2\right]I}{|\Lambda_1(\omega)|^2|\Lambda_3(\omega)|^2|\Lambda_4(\omega)|^2} + \frac{(16\Omega\kappa^2)^2I^2}{|\Lambda_1(\omega)|^4|\Lambda_3(\omega)|^2|\Lambda_4(\omega)|^2}\right]\delta(\omega+\omega')$$

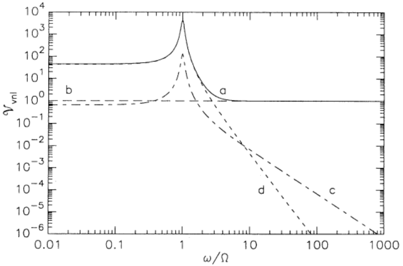


FIG. 2. (Curve *a*) The total noise, $\mathcal{V}_{vnl}$ (solid line). Contributions to $\mathcal{V}_{vnl}$: (curve *b*) photon counting noise (dashed line); (curve *c*) mirror noise (dash-dotted line); (curve *d*) radiation pressure noise (dotted line).

Trade of Between Noise Components and Optimal Power

$$h^2 = f(\omega_g)\left[\frac{1}{I} + f_1(\omega_g) + f_2(\omega_g) I\right], \tag{18}$$

where $$f(\omega) = \frac{|\Lambda_1(\omega)|^2 |\Lambda_3(\omega)|^2 |\Lambda_4(\omega)|^2}{\tau[16\omega_g^2\omega_0]^2}, \tag{19}$$

$$f_1(\omega) = \frac{16\kappa^2\gamma_b\left[(\gamma_b/2)^2 + \Omega^2 + \omega^2\right]}{|\Lambda_1(\omega)|^2 |\Lambda_3(\omega)|^2 |\Lambda_4(\omega)|^2}, \tag{20}$$

$$f_2(\omega) = \frac{(16\Omega\kappa^2)^2}{|\Lambda_1(\omega)|^4 |\Lambda_3(\omega)|^2 |\Lambda_4(\omega)|^2}. \tag{21}$$

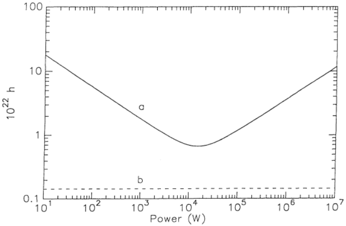
Standart Quantum Limit (SQL)

In the frequency regime in which we are interested:

$$\omega_g^2 \gg \left(\frac{\gamma_b}{2}\right)^2 + \Omega^2$$

$$h^2_{\mathsf{min}} \approx \frac{\hbar}{8M\omega_g^2 L^2 \tau \Omega}\left[2\Omega + \gamma_b\right]$$

$$h_{\mathsf{SQL}} = \frac{1}{L}\left(\frac{\hbar}{4M\omega_g^2\tau}\right)^{1/2}$$



Introducing squeezed states as input light fields

$$h_{\text{min}}^2 \approx \frac{\hbar}{8M\omega_g^2 L^2 \tau_\Omega} \left[2e^{-2|r|}\Omega + \gamma_b \right]. \tag{22}$$

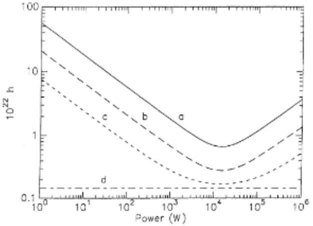


FIG. 6. The minimum possible value of h detectable as a function of power using θ^{opt}, for three different values of the squeezing parameter r: (curve a) $r=0$, (curve b) $r=1$, (curve c) $r=2$. The contribution of the mirror noise to h has also been drawn in (see curve d).

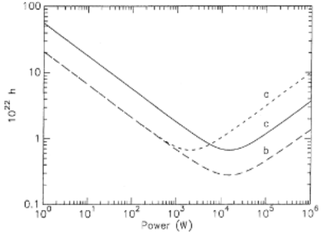


FIG. 7. A comparison between using (curve a) $\theta=0$ and (curve b) θ^{opt}, in the calculation for the minimum possible value of h detectable using $r=1$. The corresponding curve for no squeezing ($r=0$) is also shown (see curve c).

Summary

- Gravitational waves can be detected using an optical interferometer by encoding their effects in the phase relation at the output.
- The minimum detectable gravitational wave amplitude is related to the uncertainty in the amplitude, which is determined by the noise terms in the variance.
- The variance includes three noise terms, and the optimal power enabling minimum h is determined by the trade-off between these noise contributions.
- The standard quantum limit (SQL) for gravitational wave detection with optical interferometers can be surpassed by using squeezed states as input fields and incorporating a Kerr medium within the cavities.

Questions